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Extreme Lengths for Chords of an Ellipse

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Abstract

The problem originated in a Chinese university entrance practice problem. We extend the original circle case to more challenging ones for ellipses. It is evident that technological tools provide us crucial intuitions before attempting for more rigorous analytical solutions. We use GinMA [2] and ClassPad Manager [1] for simulating where the possible solutions maybe in 2D. Next, we introduce two analytical methods, one is applying the Lagrange multiplier method that involves two parameters, and the other is applying the trigonometry method using one parameter. Finally, with the help of Maple [3] as a computational engine, we verify our results using mentioned two methods indeed coincide with each other and are consistent with our geometry conjectures. The content is accessible to those who have completed the calculus courses.

1. Introduction

The problem of the paper was initiated by a university entrance practice problem in China. We consider a circle ϖ that is centered at point $O(0,0)$ of radius 2. The point $A = (1,0)$ and angle $\beta_0 = 2\beta = 90^\circ$ are given. Let B and C be two points on the circle so that the angle BAC forms a fixed angle β_0. The original problem asks Chinese students to conjecture where the extreme lengths of the chord BC on the circle should be and find the respective lengths for BC. We believe the problem is quite challenging not only for those Chinese students taking the college entrance exams but also difficult even for many university students. In this paper, we will see how technological tools can provide us crucial intuitions before attempting for more rigorous analytical solutions for circle case and when generalizing from circle case to ellipse cases. We gain geometric intuitions while using [1] or [2]. In the meantime, we use Computer Algebra System such as Maple [3] for verifying our analytical solutions with respective to various scenarios. We introduce traditional geometric approaches and the classical method of Lagrange multipliers.

We briefly state the problem for the ellipse case, we consider the following problem: We are given an ellipse ϖ(a, b), which is centered at point O(0,0), and the point A = (d,0), which can be called the observation point and is located on the major axis. In addition, the angle $\beta_0 = 2\beta$ is a given fixed angle. Let the points B and C be on the ellipse. We want to find extreme lengths of the chord BC of the ellipse. In section 5, we discuss two special cases for ellipse: One is when the observation point A is on the locus of an ellipse, and the second is to find the conditions when one end of extremal BC is in the apex of the ellipse. In section 6, we describe the scenario when we extend problem from circle to sphere. In addition, we state an open problem that is extending the extreme chords problems for ellipses to those of ellipsoids.

2. Chords for the circle case

Consider the following Figure 1(a): The observation point A = (1,0) is located on the horizontal axis of the given circle, which is centered at O(0,0) and of radius a. We denote such circle by ϖ(O, a), where $a > 1$. The angle BAC is fixed and we denote it by $\beta_0 = 2\beta$. The original Chinese university entrance practice is to conjecture where the extreme lengths of BC could be when a=2 and when the angle BAC is 90º. Before we solve this problem analytically, we use a geometry software such as [1] or [2] to conjecture where the maximum or minimum length of BC could occur. First, we move the points of B and C along the circle and observe when the length of BC will reach either maximum or minimum. Effortlessly, we discover that the maximum length occurs when BC is perpendicular to OT and BC is on the opposite side of A. Similarly, the minimum length occurs when BC is perpendicular to OT and BC is on the same site as A. Secondly, we use [1] to create the scattered plot by taking the x-component to be the x-coordinate of C and using y-component to be the length of BC as shown in Figure 1(b). By observing the scattered plot (Figure 1(b)), we may conjecture that the maximum of BC occurs when the x-coordinate of C is near the left end of the scattered plot, and the minimum of BC occurs when the x-coordinate of C is near the right end of the scattered plot, which are consistent with our earlier observations. We use the video clip [5] to demonstrate how we use [1] and [2] for making conjectures of where the maximum and minimum lengths of BC could be.

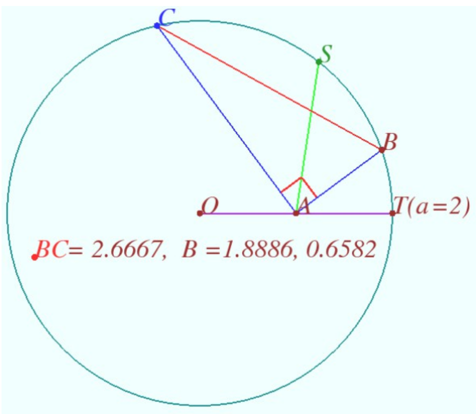


Figure 1(a). Circle with a chord

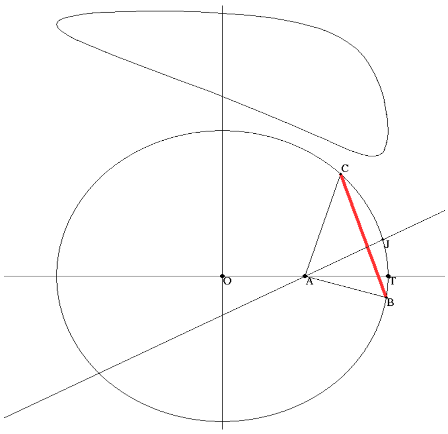


Figure 1(b). Circle and a bisector

Let us consider an analytical solution to this problem. Let the ray OA intersects the circle at the point $T(a,0)$ as shown in Figures 1(a). We introduce the variable φ as the angle of SAT, that is the angle between the bisector of the angle BAC and AT. Moreover, we consider the following two angles:

$$\varphi_1 = \angle BAT = \varphi - \beta,\ \varphi_2 = \angle CAT = \varphi + \beta. \tag{1}$$

In the triangle OBA, we use the laws of sine, and see $\dfrac{OA}{\sin OBA} = \dfrac{OB}{\sin BAO} = \dfrac{OB}{\sin BAT}$. In addition, we have

$$\angle BOA = \angle BAT - \angle OBA = \varphi - \beta - \arcsin\frac{\sin(\varphi-\beta)}{a}.$$

Similarly, $$\angle COA = \angle CAT - \angle OCA = \varphi + \beta - \arcsin\frac{\sin(\varphi+\beta)}{a}.$$

The chord BC subtends an angle θ at O, where

$$\theta = \angle COA - \angle BOA = 2\beta - \arcsin\frac{\sin(\varphi+\beta)}{a} + \arcsin\frac{\sin(\varphi-\beta)}{a}. \tag{2}$$

Since the length of the chord $BC = 2a\sin\frac{\theta}{2}$ changes monotonically over the angle θ, the extremum of BC corresponds to the extremum of the expression:

$$f(\varphi) = \arcsin\frac{\sin(\varphi+\beta)}{a} - \arcsin\frac{\sin(\varphi-\beta)}{a}. \tag{3}$$

Neglecting a positive factor, we obtain:

$$f'(\varphi) \sim \cos(\varphi+\beta)\cdot\sqrt{a^2-1+\cos^2(\varphi-\beta)} - \cos(\varphi-\beta)\cdot\sqrt{a^2-1+\cos^2(\varphi+\beta)}.$$

We note that if $\varphi = 0$ or $\varphi = \pi$, then $f'(\pi) = 0$, these are the angles at which f will have extreme value. For other cases such as when $\varphi\in(0,\pi)\cap\beta\in(0,\frac{\pi}{2})\Rightarrow\cos(\varphi+\beta)<\cos(\varphi-\beta)\Rightarrow f'(\varphi)<0$, there are no other extreme points. Therefore, we have the following observations:

$$\text{If } \varphi = 0,\quad \theta = 2\beta - 2\arcsin\frac{\sin\beta}{a}.\qquad BC_{min} = 2a\sin\frac{\theta}{2} = 2a\sin(\beta - \arcsin\frac{\sin\beta}{a}). \tag{4}$$

$$\text{If } \varphi = \pi,\quad \theta = 2\beta + 2\arcsin\frac{\sin\beta}{a}.\qquad BC_{max} = 2a\sin\frac{\theta}{2} = 2a\sin(\beta + \arcsin\frac{\sin\beta}{a}). \tag{5}$$

The Figure 2 shows the graphs of $f(\varphi, \beta_0 = 90^o)$ and the Figure 3 shows the graph of $f'(\varphi, \beta_0 = 90^o)$ for $a = 4$ (red), $a = 2$ (blue) and $a = 1.25$ (green), respectively. It is easy to see that since the factor a is only a scaling factor, the graphs in red, blue and green in Figures 2 and 3 are all similar.

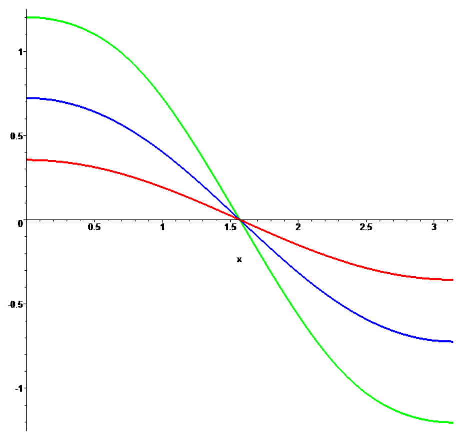


Figure 2. Graph of $f(\varphi, \beta_0 = 90^o)$

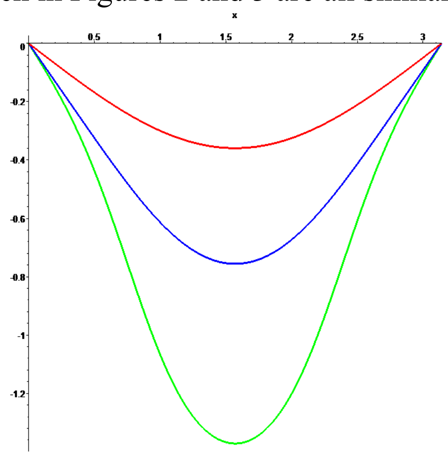


Figure 3. Graph of $f'(\varphi, \beta_0 = 90^o)$

Remarks:

(a) Both Figures 2 and 3 show that extreme values for BC occur at $\varphi = 0$ and $\varphi = \pi$ respectively.

(b) The method we describe here, involving traditional trigonometry and calculus techniques, is definitely non-trivial to some university students. It will be even more challenging to those high school students who are preparing for the college entrance exam in China.

(c) We leave to readers to explore and verify that when we modify the angle for β_0 so that the angle β_0 is between 0 and π, the extreme values for BC remain at $\varphi = 0$ and $\varphi = \pi$ respectively.

3. Chords for the ellipse case

Now we extend the problem to ellipses. Let the observation point $A = (d, 0)$, where $d < 1$, be inside the given ellipse, that is centered in $O(0,0)$ with major and minor axes a and b respectively and $a > b \geq 1$. We denote such ellipse by $\omega(O, a, b)$. Let B and C be on the ellipse so that the angle BAC is a fixed angle. In other words, the angle BAC subtended by a chord of the ellipse BC is fixed at $\beta_0 = 2\beta$. We want to find the extreme values of the chord length BC. As we have done for the circle case, we first make use of [1] and [2] to simulate or conjecture where the solutions might be. We draw the given ellipse and point A. We place moving point B on the ellipse curve, rotate AB by a fixed angle β_0 around point A, construct ray AC and define the point C as the point of the intersection of ellipse and ray AC (See Figure 4 or 5). Before solving this problem analytically, we again emphasize that the tools of [1], [2] or similar allow us to make challenging problems more accessible to general audience. For example, we may place the point B at any position on the ellipse and record the length of the chord BC (see Figure 4). The green curve in the Figure 5 shows the scattered plot when we use the x-coordinates of the point C as inputs (x) and the lengths of BC as the outputs (y). Learners can conjecture where the positions of C will correspond to the extreme lengths of BC by moving the point C. We refer readers to video clip [6] to see how we use [1] and [2] for making conjectures of where the maximum and minimum lengths of BC could be.

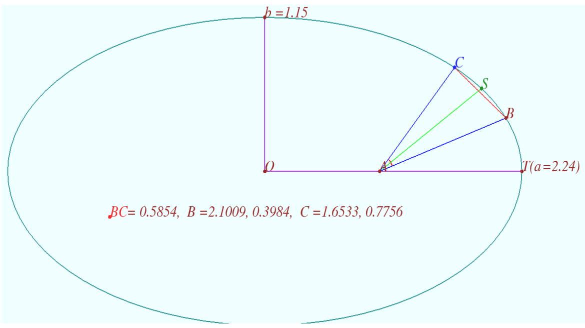


Figure 4. GInMA simulation

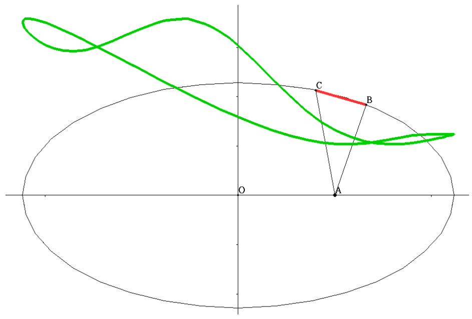


Figure 5. ClassPad Manager simulation

Now we describe how we use the following two methods to solve this problem analytically.

Method 1. The two-parameters Lagrange method. We introduce a pair of variables angles subtended by the ends of the chord from the center of the ellipse. Consider the following Figure 6:

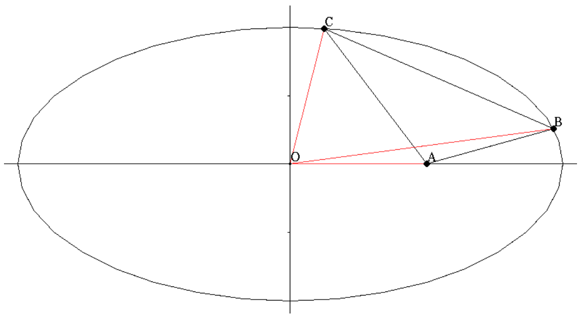


Figure 6. Method 1 with two parameters

We set $\phi_1 = \angle BOA, \phi_2 = \angle COA$. Then the coordinates of points for B and C can be written as

$$x_B = a\cos\phi_1,\ y_B = b\sin\phi_1,\ x_C = a\cos\phi_2,\ y_C = b\sin\phi_2,$$

respectively. We consider $BC^2 = a^2(\cos\phi_1 - \cos\phi_2)^2 + b^2(\sin\phi_1 - \sin\phi_2)^2$,

hence may consider the following objective equation

$$f_1(\phi_1, \phi_2) = a^2(\cos\phi_1 - \cos\phi_2)^2 - b^2(\sin\phi_1 - \sin\phi_2)^2 = 0. \tag{6}$$

On the other hand, we note that

$$\vec{AB}=(x_B-x_A,\, y_B)=(a\cos\phi_1-d,\, b\sin\phi_1),\ \vec{AC}=(a\cos\phi_2-d,\, b\sin\phi_2),$$

and the angle angle between $\vec{AB}$ and $\vec{AC}$ is β_0. Therefore, we have $\cos\beta_0=\dfrac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}||\vec{AC}|}$, and hence we consider the following constraint equation:

$$f_2(\phi_1,\phi_2)=\cos\beta_0-\frac{(a\cos\phi_1-d)\cdot(a\cos\phi_2-d)+b^2\sin\phi_1\cdot\sin\phi_2}{\sqrt{(a\cos\phi_1-d)^2+b^2\sin^2\phi_1}\cdot\sqrt{(a\cos\phi_2-d)^2+b^2\sin^2\phi_2}}=0. \tag{7}$$

To find extreme lengths of the chord we set up the Lagrange multiplier function as follows:

$$F(\phi_1,\phi_2,\lambda)=f_1(\phi_1,\phi_2)+\lambda f_2(\phi_1,\phi_2). \tag{8}$$

Therefore, the problem is reduced to solve a system of followings equations by setting the partial derivatives of F equal to 0. In other words, we consider the followings:

$$\begin{cases} F'_{\phi_1}(\phi_1,\phi_2,\lambda)=0, \\ F'_{\phi_2}(\phi_1,\phi_2,\lambda)=0, \\ F'_{\lambda}(\phi_1,\phi_2,\lambda)=0, \end{cases} \Rightarrow \begin{cases} f_1{}'_{\phi_1}(\phi_1,\phi_2)+\lambda f_2{}'_{\phi_1}(\phi_1,\phi_2)=0, \\ f_1{}'_{\phi_2}(\phi_1,\phi_2)+\lambda f_2{}'_{\phi_2}(\phi_1,\phi_2)=0, \\ f_2(\phi_1,\phi_2)=0. \end{cases} \tag{9}$$

With the help of CAS [3], we can find appropriate solutions of ϕ_1,ϕ_2 and λ, which yield to maximum and minimum lengths of BC respectively, see Maple worksheet [4].

<u>*Method 2. A one-parameter trigonometry representation.*</u> We introduce φ as the variable angle between the bisector of the angle *BAC (SA)* and the axis of the ellipse *OT,* where $T=(a, 0)$, see Figure 7. We further consider the following two new angles: $\varphi_1=\angle BAT=\varphi-\beta$, and $\varphi_2=\angle CAT=\varphi+\beta$.

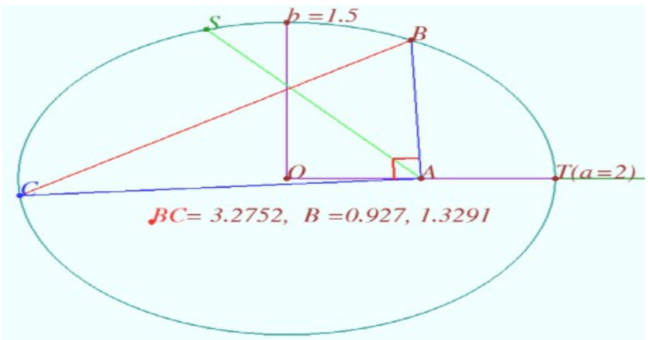


Figure 7. Method 2 with one parameter

We note that the coordinates for the points B and C can be found by using the following equations:

$$x_B=d+t\,y_B,\ y_B=\begin{cases} y_{B_1},\ \varphi\le\beta, \\ y_{B_2},\ \beta<\varphi\le\pi+\beta, \\ y_{B_1},\ \varphi>\pi+\beta, \end{cases} \quad x_C=d+t_2\,y_C,\ y_C=\begin{cases} y_{C_1},\ \varphi\le\pi-\beta, \\ y_{C_2},\ \pi-\beta<\varphi\le 2\pi-\beta, \\ y_{C_1},\ \varphi>2\pi-\beta, \end{cases}$$

where $y_{B_1}=b\dfrac{-bdt+a\sqrt{b^2t^2-d^2+a^2}}{b^2t^2+a^2},\ y_{B_2}=b\dfrac{-bdt-a\sqrt{b^2t^2-d^2+a^2}}{b^2t^2+a^2},\ t=\cot(\varphi-\beta),$

$$y_{C_1}=b\frac{-bdt_2+a\sqrt{b^2t_2^2-d^2+a^2}}{b^2t_2^2+a^2},\ y_{C_2}=b\frac{-bdt_2-a\sqrt{b^2t_2^2-d^2+a^2}}{b^2t_2^2+a^2},\ t_2=\cot(\varphi+\beta). \tag{10}$$

The square length of BC is therefore

$$BC^2 = (x_B - x_C)^2 + (y_B - y_C)^2. \qquad (11)$$

Since the length of the chord BC^2 is a function of φ, which is expressed in equation (11), so the problem of finding the extremum of BC^2 is reduced to solve the following equation:

$$\frac{d\left(BC^2\right)}{d\varphi} = 0. \qquad (12)$$

We refer readers to the Maple worksheet [4] for computations when using Method 2.

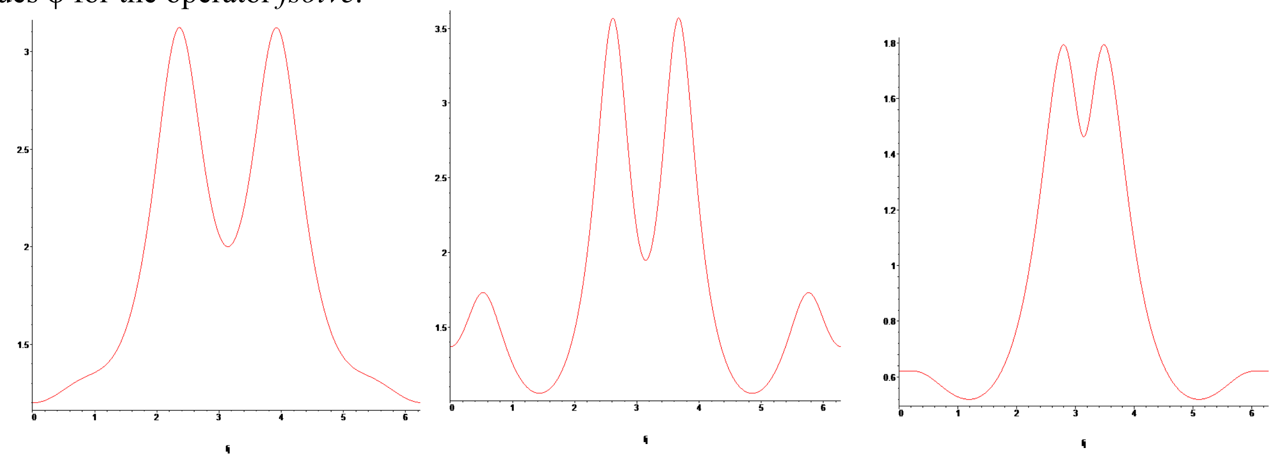
Remarks.

(1) The advantage of using Lagrange method is that the problem becomes clear once the objective function and subjection functions are set up correctly. Of course, students would have to wait until they learn this method in their multivariable calculus semester. On the other hand, method 2 of using one variable can be an excellent exercise for students, as an application of derivative, in their first semester calculus course. The challenging part for students is being able to express the respective x and y coordinates for the points B and C before setting the square length function of BC.

(2) We note that we use the approach of expressing the length as a function f and finding the zeros of its derivative f' by solving $f'=0$ numerically with Maple. Readers, of course, may use some other efficient numerical procedures for optimizing a function using any CAS.

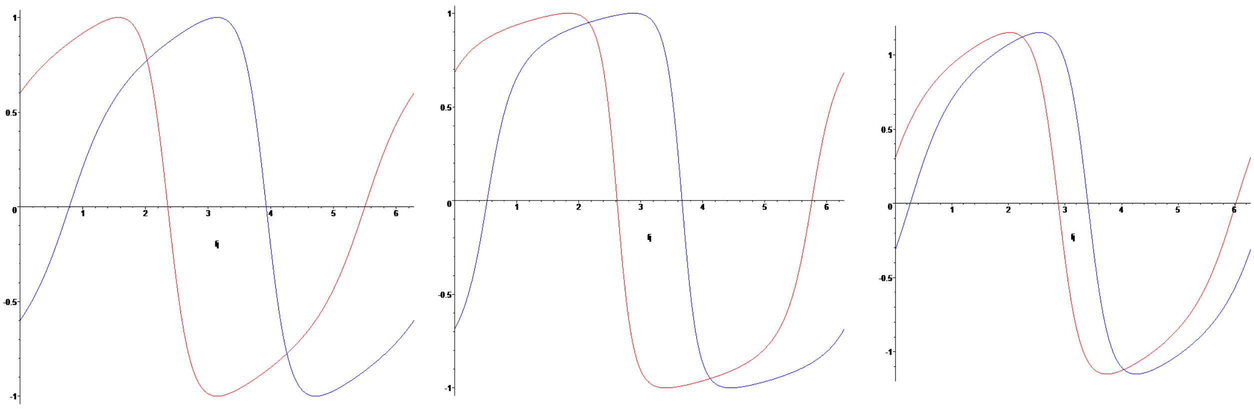
4. Examples and numerical solutions for ellipses

For demonstration purpose, we choose the following three cases:
Case 1: The parameters (a, b, d, β_0) are (2,1,1,90º),
Case 2: The parameters (a, b, d, β_0) are (3,1,1,60º),
Case 3: The parameters (a, b, d, β_0) are (2.24,1.15,1,30º).
We refer readers to Maple worksheet [4] and note the following observations:
(A) When we apply the two-parameter Lagrange method with *Maple,* we use the operator *fsolve*. We get sequence of solutions using the options *avoid*.
(B) When we apply the one-parameter method with Maple, we use the operator *fsolve*. We try to build a whole sequence of solutions using options *avoid*; however, Maple can find only 2 or 3 solutions. Therefore, we plot the function $BC(\varphi)$ as shown in Figures 8(a)-(c). We use plots to find the approximate values of φ that correspond to extreme values for $BC,$ and give *Maple* a range of values φ for the operator *fsolve*.



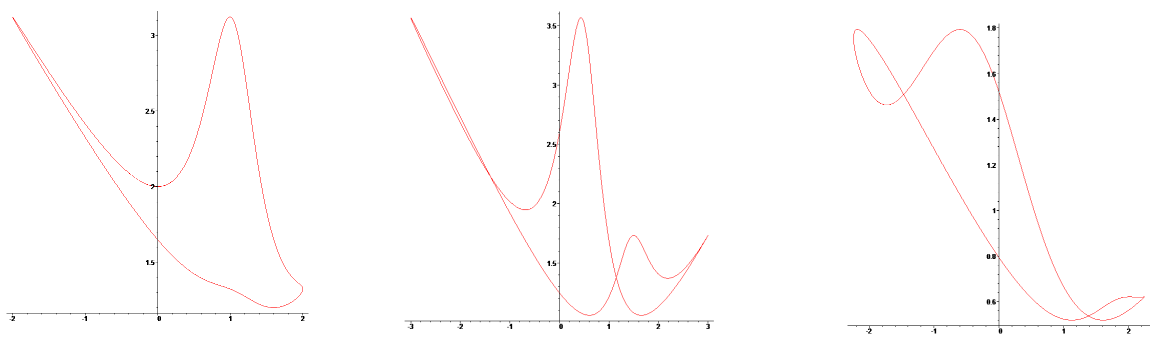
Figures 8(a), (b) and (c). Function $BC(\varphi)$ for test cases

The derivation of formula for $y(\varphi)$, by selecting the correct solutions, are described in equation (10) on the respective intervals. The graphs for $y(\varphi)$ at respective test cases are shown in Figures 9(a)-(c).



Figures 9(a), (b) and (c). Dependence $y(\varphi)$ for test cases

The graphs for the function $BC(x_B)$ are shown in Figures 10(a)-(c) for respective test cases.



Figures 10(a), (b) and (c). Lengths of BC as function of x_B

The following displays the screen shots of solutions for our test case 1 while applying Method 1 of two variables:

"Ellipse a = ", 2, " b = ", 1, " d = ", 1, " beta0 = ", $\frac{\pi}{2}$

"phi = ", 2.361081791, " BC = ", 3.122668411, " B=(", 0.9957614769, 0.8672455075, 1.049642940, ")", " C=(", -1.999785046, -0.01466096741, -0.01466149268, ")"

"phi = ", 0, " BC = ", 1.200000000 , " B=(", 1.600000000 , -0.6000000000 , ")", " C=(", 1.600000000 , 0.6000000000 , ")"

"phi = ", π, " BC = ", 2.000000000 , " B=(", -0.2 10^{-8}, 1.000000000 , ")", " C=(", -0.2 10^{-8}, -1.000000000 , ")"

"phi = {", 0.01466149443 , -7.332828247 , "}, BC = ", 3.122668409 , " B = {", -2.999785044 , 0.01466096916 , "}"

"phi = {", -8.781276852 , -5.639684198 , "}, BC = ", 1.200000000 , " B = {", 0.600000000 , -0.6000000000 , "}"

"phi = {", -4.712388980 , -1.570796327 , "}, BC = ", 2.000000000 , " B = {", -0.9999999992 , 1., "}"

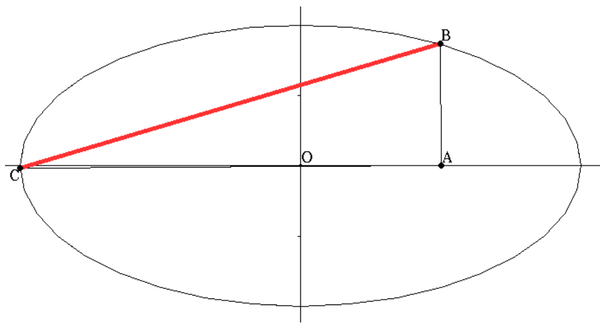
The followings displays the screen shots of solutions for our test case 1 while applying Method 2.

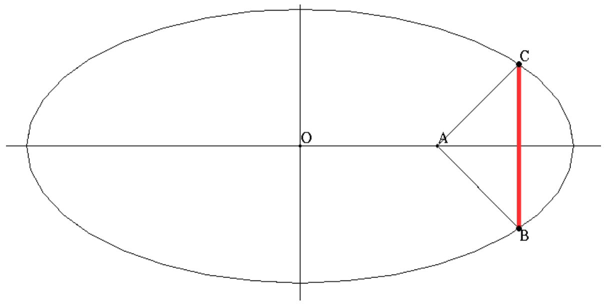
"phi = ", 0, " BC = ", 1.200000000, " B=(", 1.600000000, −0.6000000000, ")", " C=(", 1.600000000, 0.6000000000, ")" "phi = ", π, "   BC = ", 2.000000000, "   B=(", $-2.\ 10^{-9}$, 1.000000000, ")", "   C=(", $-2.\ 10^{-9}$, -1.000000000, ")"

We use *avoid* command within Maple to extract only three solutions from the original five. We note from the numeric computations with Maple that both methods give identical results up to the last significant digit. We summarize the extreme lengths for *BC* in our respective cases as follows:

Case 1: The parameters (a, b, d, β_0) are (2,1,1,90º)

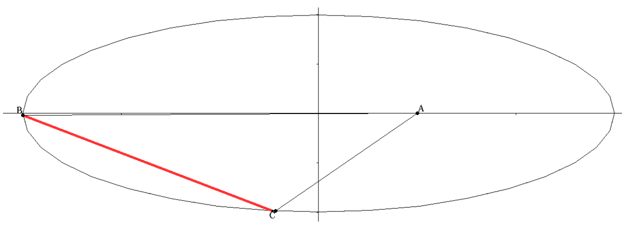
Maximum Length *BC = 3.122668411* *B=[0.9957614769,0.8672455075],* *C=[-1.999785046,-0.01466096741*	*Minimum BC = 1.200000000,* *B=[1.600000000,-0.6000000000],* *C=[1.600000000,0.6000000000]*

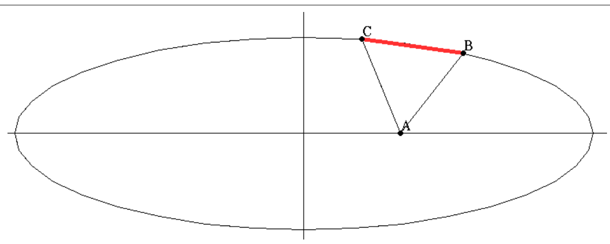




Case 2: The parameters (a, b, d, β_0) are (3,1,1,60º)

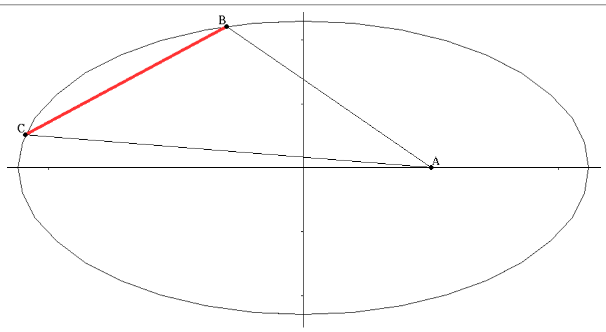
Maximum Length *BC=3.568936526,* *B = [-2.999587153,-0.01658952556],* *C=[-0.4341875895,-0.9894713031*	*Minimum BC = 1.057839712,* *B = [1.652679536,0.8345757640],* *C=[0.6048096464,0.9794672754]*

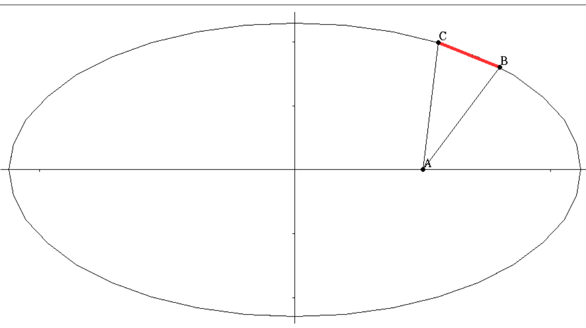




Case 3: The parameters (a, b, d, β_0) are (2.24,1.15,1,30º)

Maximum Length *BC=1.793487504,* *B=[-0.601424782,1.107773814],* *C=[-2.181950627,0.2601090923]*	*Minimum BC=0.5195700851,* *B=[1.605253013,0.8020706171,* *C=[1.122862394,0.9950793325]*





5. Special Cases for Ellipses

We discuss two special cases for this extreme lengths of chord for the ellipse cases as follows:

5.1. When *A* is a focus point of the ellipse

Let the given ellipse be centered in $O(0,0)$ with major and minor axes a and b respectively, and $a > b \geq 1$. In addition, we write $T = (a, 0)$, $A = F = (d, 0)$, where F is a focus point for the ellipse. We pick B and C to be on the ellipse and note that $d=\sqrt{a^2-b^2}, e=\frac{d}{a}$. The angle of BFC or BAC forms a fixed angle $\beta_0 = 2\beta\in[0, 150°]$. We want to find the extreme lengths of BC.

We refer to Figure

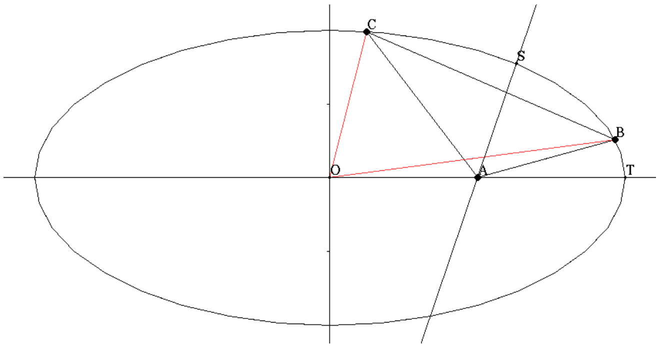


Figure 11. A special case when point *A* is the locus

We set $\angle CAT = \varphi$ as before and observe the followings:

$$\angle TAB = \varphi - \beta \text{ and } \angle TAC = \varphi + \beta.$$

If we set $p=b^2/a$, we may write

$$AB=\frac{p}{1+e\cos(\varphi-\beta)}, AC=\frac{p}{1+e\cos(\varphi+\beta)}. \tag{13}$$

Next we set

$$u=1+e\cos\varphi\cos\beta, v=e\sin\varphi\sin\beta. \tag{14}$$

Then

$$AB=\frac{p}{u+v}, AC=\frac{p}{u-v}. \tag{15}$$

This implies that

$$BC^2=AB^2+AC^2-2\cdot AB\cdot AC\cos 2\beta, \frac{BC^2}{4p^2}=\frac{u^2\sin^2\beta+v^2\cos^2\beta}{(u^2-v^2)^2}. \tag{16}$$

By taking $\frac{d}{d\varphi}\left(\frac{BC^2}{4p^2}\right)=0,$ we see

$$\frac{d}{d\varphi}\left(\frac{BC^2}{4p^2}\right)=0\Rightarrow(u^2\sin^2\beta+v^2\cos^2\beta)'\cdot(u^2-v^2)=2(u^2-v^2)'\cdot(u^2\sin^2\beta+v^2\cos^2\beta). \tag{17}$$

We make transformation and get two solutions:

$$\sin\varphi = 0$$

and

$$3e^2\cos^2\varphi+2e\cos\varphi\left(\frac{1}{\cos\beta}+\cos\beta+e^2\cos\beta\right)+1+e^2(1+\cos^2\beta)=0.$$

We extract the root and obtain

$$3e\cos\varphi = \pm\sqrt{det} - \left(\frac{1}{\cos\beta} + \cos\beta + e^2\cos\beta\right) + 1 + e^2(1+\cos^2\beta) = 0, \tag{18}$$

where

$$det = \frac{1}{\cos^2\beta} - 1 + \cos^2\beta - e^2(1+\cos^2\beta) + e^4\cos^2\beta.$$

Interesting case is when

$$a=3, b=\sqrt{5}, d=2, 2\beta=60^\circ \Rightarrow \varphi_{min}=0, \varphi_{max}=150^\circ.$$

For other cases, answer are not so simple. For example, if we use $a=3, b=1, d=2\sqrt{2}, 2\beta=90$, we obtain the following results:

$$\cos\varphi_{min} = \frac{35+\sqrt{91}}{36}, \cos\varphi_{max} = \frac{35-\sqrt{91}}{36}.$$

5.2. When extreme end of the chord is located at the vertex (-*a*, 0) of the ellipse

We consider the case when one end of the longest chord is situated at the vertex (-*a*, 0) of the ellipse and the observation point A is a focus point of the ellipse. Then we observe from Figure 12 that

$$\varphi + \beta = \pi \Rightarrow \cos\varphi = -\cos\beta. \tag{19}$$

We substitute (19) into (18) and obtain

$$\cos^2\beta = \frac{(e-1)^2}{2e(e-1)^2} = \frac{1}{2e} = \frac{a}{2d}.$$

For example, if $\beta_0 = 2\beta = 60^\circ$, then:

$$\cos^2\beta = \frac{3}{4} \Rightarrow \frac{3}{2}d = a.$$

Since $e < 1$, then this is possible only if $\cos\beta > \frac{1}{\sqrt{2}}, \beta < \frac{\pi}{4}$.

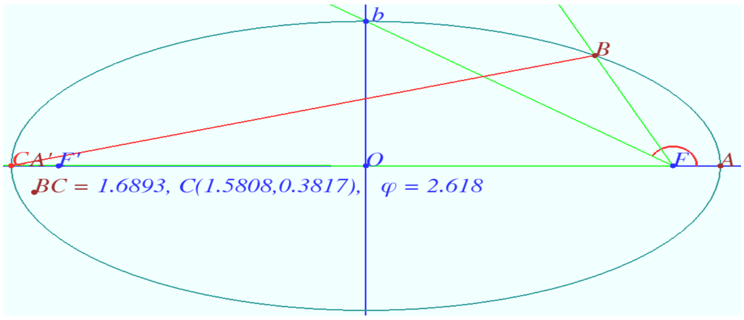


Figure 12. One end of the longest chord is situated at the vertex

6. Extensions to sphere case and an open problem

It is not difficult to generalize this problem from a circle to a sphere in 3D. We will not solve the 3D scenario analytically. Instead, we describe how the software [2] gives a rich visual intuition to our conjectures. We describe the 3D problem as follows: Given a sphere of a fixed radius r of the form $x^2+y^2+z^2=r^2$ and pick the point $A=(d,0,0)$, where $d\in[0,r]$. Let B be a point on the sphere and rotate AB with a fixed angle β to form a cone, see Figure 13. We want to find, respectively, the maximum and minimum intersecting surface areas between the cone and the sphere.

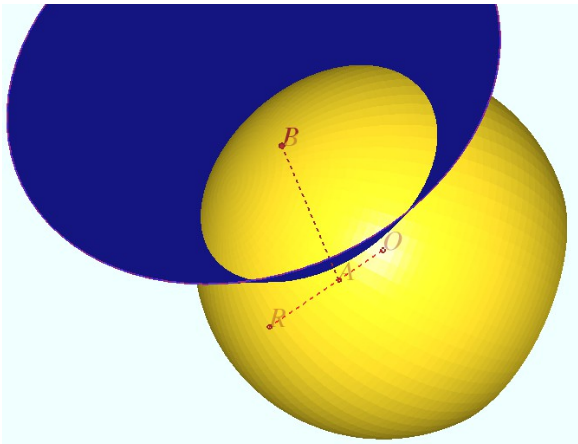


Figure 13. Sphere case

After exploring with [2], which we refer readers to the clip [6], it is not difficult to see that the maximum intersecting surface area occurs when the normal vector at *B* is parallel to the vector *OA* and *A* is on the opposite of *B* (see Figure 14(a)). The minimum intersecting surface area occurs when the normal vector at *B* is parallel to the vector *OA* but *A* is on the same side of *B* (see Figure 14(b)).

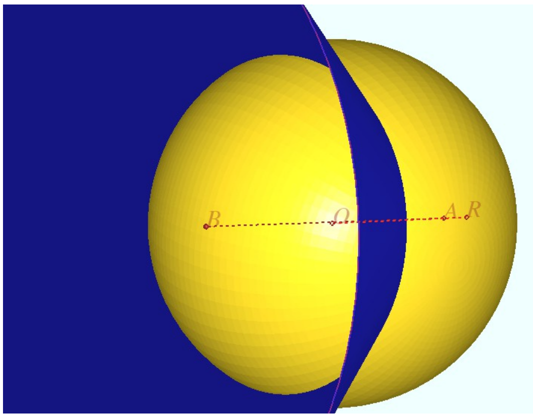


Figure 14(a). Maximum Surface Area

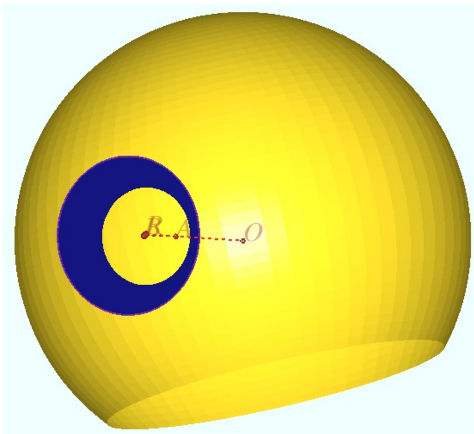


Figure 14(b). Minimum Surface Area

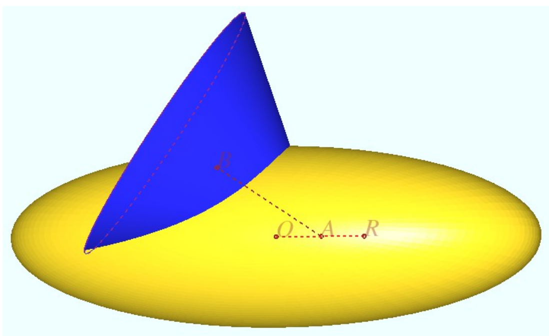


Figure 15 An ellipsoid and a cone

The problem becomes even more challenging if we extend the ellipse to an ellipsoid. For example, given an ellipsoid of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, let $A=(x,0,0)$ with $x\in[0,a]$. Pick *B* be on the ellipsoid and consider the cone that is determined by rotating the axis *AB* with a fixed angle β, see Figure 15. We therefore describe the following

O***pen problem*****:** For a fixed point A and fixed angle for the cone, find the point B that will result in the maximum or minimum intersecting surface area between the ellipsoid and the cone (see Figure 15). Just like the case for ellipse, the answers vary depending on the values of a, b, c, the fixed angle β and the position of the point A. The video clip of this ellipsoid and a cone problem can be seen in the latter part of [6].

7. Conclusion

It would have been a non-trivial task if the problem for circle case was given at a real college entrance exam in China, where only paper and pencil are allowed. Normally, students are allocated no more than 10 minutes to solve one problem. Under such circumstance, it is not hard to predict that many students would have lost interests in attempting to solve this problem. It is evident that technological tools can provide us crucial intuitions before we attempt for more rigorous analytical solutions. Here we gain geometric intuitions while using [1] and [2]. In the meantime, we use the Computer Algebra System, Maple [3], for verifying our analytical solutions are consistent with our conjectures inspired by using [1] and [2]. Therefore, without the help of a geometric software, it is definitely challenging to even conjecture the solutions for extreme lengths for the circle case. Evolving technological tools definitely have made mathematics fun and accessible on one hand, but they have made mathematics more challenging and theoretical at the same time. We hope when mathematics is made more accessible to students, it is possible more students will be inspired in investigating problems from simple case to more challenging ones. We do not expect the exam-oriented curricula will change overnight for many countries. However, how to make more students be interested in mathematics with the help of advancing technological tools is an important task for many educators.

References:

[1] Classpad Manager, a product of CASIO, https://edu.casio.com/freetrial/freetrial_list.php
[2] Geometry Interactive Mathematical Arts (GInMA): A Dynamic Geometry System, see http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx.
[3] Maple: A product of Maplesoft, see http://maplesoft.com/.

Supplementary Electronic Materials:

[4] A Maple worksheet: http://www.radford.edu/wyang/eJMT_June2015/Maple.mws.
[5] A video clip for a 2D circle conjecture: https://mathandtech.org/eJMT_June2015/circle_conjecture.mp4.
[6] A video clip for a 3D sphere, 2D ellipses and 3D ellipsoid conjecture: https://mathandtech.org/eJMT_June2015/sphere.mp4.